LECTURE 24 LOCAL EXTREMA AND CRITICAL POINTS

Finding extremas is one necessary step to determine the optimal solutions. First, we have a necessary condition for local extremas.

Theorem. *(First derivative theorem) If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.*

Remark. Note that the point $x = d$ in the figure from last lecture above does not fall in the hypothesis of the theorem, even though it is a local maximum. Note that the derivative $f'(c)$ is undefined since it is kink. Therefore, when one wants to use the theorem here, one must be sure that the derivative is defined at the point of interest, in addition to it being a local extrema.

Theorem. *(Contrapositive statement of the First derivative theorem, namely, "if not B then not A".)*
If $f'(c)$ is defined but $f'(c) \neq 0$ where c is an interior point of the domain of f, then f cannot achieve a local maximum or minimum at $x = c$.

However, the theorem does not tell us how to sufficiently qualify a point as a local extrema. It only tells us, if it is one, then it necessarily has zero slope. Is the converse true? Apparently not.

Example. ($f'(c) = 0$ does not imply that $x = c$ is a local max or min) Consider the function $f(x) = x^3$. We find that $f'(x) = 3x^2$, which means $f'(0) = 0$. We learned from recitation that $x = 0$ is neither a local max nor a local min, since any neighbourhoods containing $x = 0$ will have some x-values that have either larger or smaller function values than 0.

But the points that achieve $f'(c) = 0$ are still valuable in the sense that they COULD be local maximum or minimum. In fact, there are three possible places where extreme values can be achieved (global or local):

(1) Interior points where $f' = 0$.
(2) Interior points where f' is undefined.
(3) Endpoints of the domain f.

Definition. An interior point of the domain of a function f where f' is **zero** or **undefined** is a **critical point** of f.

Remark. Note that being a critical point does NOT guarantee that it is an extreme value.

The above definitions and theorems give us a procedure to find the global extreme values of f on a given interval $[a, b]$.

(1) Find all critical points of f on the interval.
(2) Evaluate f at the critical points and endpoints. (think about why the endpoints could be absolute max or min)
(3) Largest value is global max, while smallest value is global min.

Example. Find the absolute max and min of $g(t) = 8t - t^4$ on $[-2, 1]$.

Solution. We follow the strategy as above.

(1) Critical points.
First, we find $g'(t) = 8 - 4t^3$. Setting it equal to 0, we find a critical point that satisfy
$$0 = 8 - 4t^3 \implies t^3 = 2 \implies t = 2^{\frac{1}{3}}.$$
However, note that $2^{\frac{1}{3}} > 1$, which means this critical point is outside our interval $[-2, 1]$. Therefore, there exists NO critical points in $[-2, 1]$.
(2) Evaluation of the critical points and the endpoints.
We find that the endpoints are $t = -2$ and $t = 1$.
$$g(-2) = 8 \cdot (-2) - (-2)^4 = -16 - 16 = -32$$
$$g(1) = 8 \cdot (1) - (1)^4 = 7$$

(3) Compare the values found in step 2.
We find that $(1,7)$ is an absolute maximum, and $(-2,-32)$ an absolute minimum.

Remark. Since $g'(t) \neq 0$ on $[-2,1]$, this implies that neither local max nor local min exists on $[-2,1]$, per the contrapositive statement of the first derivative theorem.

Example. Identify the domain of $y = f(x) = x^{2/3}(x+2)$ and find its local max and min.

Solution. The domain is $(-\infty, \infty)$.

We find the critical points (suspect local extrema). First,

$$y' = \frac{2(x+2)}{3x^{\frac{1}{3}}} + x^{\frac{2}{3}} = \frac{2(x+2)}{3x^{\frac{1}{3}}} + \frac{x}{x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}}\left(\frac{2}{3}(x+2)+x\right).$$

Setting the derivative equal to 0, we find critical points that satisfy

$$\frac{2}{3}(x+2)+x = 0 \implies x = -\frac{4}{5}.$$

Also, since critical points also include the points such that y' is undefined, then $x=0$ is another one.

Now, $y\left(-\frac{4}{5}\right) = \left(-\frac{4}{5}\right)^{\frac{2}{3}}\left(-\frac{4}{5}+2\right) = \left(\frac{16}{25}\right)^{\frac{1}{3}}\left(\frac{6}{5}\right) = \left(\frac{2^3\cdot 2\cdot 5}{5^3}\right)^{\frac{1}{3}}\left(\frac{6}{5}\right) = \frac{2}{5}\cdot\frac{6}{5}\cdot 10^{\frac{1}{3}} = \frac{12}{25}\cdot 10^{\frac{1}{3}}$. Anyways, besides the windy simplification steps here, $y\left(-\frac{4}{5}\right) > 0$. By the first derivative theorem, we learn that if a point is a local extrema, then it is necessarily a critical point, i.e. local extrema can only happen critical points (on the interior of the domain, by definition of critical point, so no need to think about endpoints).

At the same time, $y(0) = 0$. This point is a local minimum.

More systematically, we form the following table on the number line.

Critical points/intervals	$x < -\frac{4}{5}$	$x = -\frac{4}{5}$	$-\frac{4}{5} < x < 0$	$x = 0$	$x > 0$
$f'(x)$	$+$	0	$-$	undefined	$+$
$f(x)$	increasing	local max	decreasing	local min	increasing

To check the sign of the first derivative on various intervals, we plug in a typical values in that interval (hopefully convenient for evaluation), and find out the sign. If it is positive, it means that the function is increasing, and decreasing otherwise. When a function increases up to a critical point, and decreases past it, then the critical point is a local max. Similarly, if a function decreases down to a critical point, and increases past it, then the critical point is a local min.